$$C(K, S_t, \tau, \sigma_t, r) = e^{-r\tau} \int_K^{\infty} (S_T - K)^+ f(S_T | S_t, \tau) dQ_\tau(S_T). \tag{1}$$

$$P(K, S_t, \tau, \sigma_t, r) = e^{-r\tau} \int_0^{K} (K - S_T)^+ f(S_T | S_t, \tau) dQ_\tau(S_T), \tag{2}$$

$$m = (K/S_t)^{\Psi}, \tag{3}$$

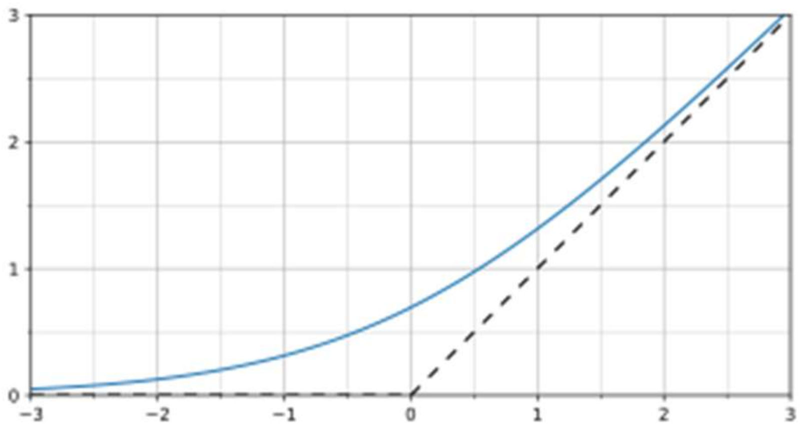
where Ψ is a call/put indicator: it is 1 for call options and -1 for put options. This design accommodates both call and put options with $m = K/S_t$ and $m = S_t/K$, respectively.

$$y(m, \tau, \sigma_t, r) = \sum_{j=1}^{N_h} \left[\sigma_+(b_j^m - me^{w_j^m})\right]\left[\sigma_+(b_j^\tau + \tau e^{w_j^\tau})\right] \times \left[\sigma_+(b_j^r \pm re^{w_j^r})\right]\left[\sigma_+(b_j^{\sigma_t} + \sigma_t e^{w_j^{\sigma_t}})\right], \tag{4}$$

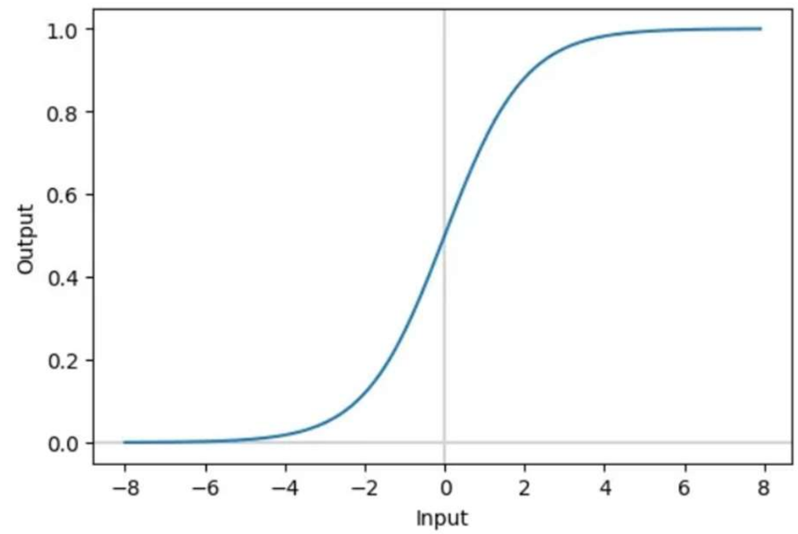
where $\sigma_+()$ is the *softplus* function $\sigma_+(x) = \log(1 + e^x)$. The weights ($w_j^m$, w_j^τ, w_j^r, $w_j^{\sigma_t}$) and biases (b_j^m, b_j^τ, b_j^r, $b_j^{\sigma_t}$) are parameters to be estimated. The + and - in $(b_j^r \pm re^{w_j^r})$ are for call and put options, respectively. The sign in each $\sigma_+()$ function is designed according to specific constraints.

Softmax

$\log(1 + e^{x})$



Sigmoid



(c1) *Convexity in K*

Both C and P are convex across K for $\tau \geq 0$. C is monotonically non-increasing with K, whereas P is monotonically non-decreasing with K. Hence, $\frac{\partial C}{\partial K} \leq 0$ and $\frac{\partial P}{\partial K} \geq 0$.

Proof: Constraint (c1). The derivative of a *softplus* function $\sigma_+(x)$ can be obtained as follows:

$$\frac{d\log(1+e^x)}{dx} = \frac{e^x}{1+e^x} = \frac{1}{1+e^{-x}}. \tag{5}$$

The function $\frac{1}{1+e^{-x}}$ is called the *sigmoid*, which can also be used as an activation function. We represent it as $\sigma_s = \frac{1}{1+e^{-x}}$ thus $\sigma'_+(x) = \sigma_s(x)$. In this way, constraint (c1) can be written as follows:

$$\frac{\partial y}{\partial m} = \sum_{j=1}^{N_h} \underline{\left[-e^{w_j^m}\sigma_s(b_j^m - me^{w_j^m})\right]}\left[\sigma_+(b_j^\tau - \tau e^{w_j^\tau})\right]\left[\sigma_+(b_j^r \pm re^{w_j^r})\right] \times \left[\sigma_+(b_j^{\sigma_t} + \sigma_t e^{w_j^{\sigma_t}})\right]. \tag{6}$$

$$f(m) = b_j^m - me^{w_j^m}$$

$$\frac{\partial}{\partial m}\sigma_+(f(m)) = \sigma'_+(f(m)) \cdot f'(m)$$

$$= \sigma_s(b_j^m - me^{w_j^m}) \cdot (-e^{w_j^m})$$

Hence, $\frac{\partial y}{\partial m} \leq 0$. Consider the definition of moneyness, we have the following for call options:

$$\frac{\partial y}{\partial K} = \frac{\partial y}{\partial m}\frac{\partial m}{\partial K} = \frac{\partial y}{\partial m}\frac{1}{S_t} \leq 0. \tag{7}$$

Likewise for put options:

$$\frac{\partial y}{\partial K} = \frac{\partial y}{\partial m}\frac{\partial m}{\partial K} = -\frac{\partial y}{\partial m}\frac{S_t}{K^2} \geq 0. \tag{8}$$

□

(c2) *Monotonicity in* τ

Both C and P are non-decreasing with $K > 0$. Hence, $\frac{\partial C}{\partial \tau} \geq 0$ and $\frac{\partial P}{\partial \tau} \geq 0$.

Proof: Constraint (c2). Similarly, we can express the constraint (c2) for call and put options as follows:

$$\frac{\partial y}{\partial \tau} = \sum_{j=1}^{N_h} \left[\sigma_+(b_j^m - me^{w_j^m})\right]\left[e^{w_j^\tau}\sigma_s(b_j^\tau + \tau e^{w_j^\tau})\right]\left[\sigma_+(b_j^r \pm re^{w_j^r})\right] \times \left[\sigma_+(b_j^{\sigma_t} + \sigma_t e^{w_j^{\sigma_t}})\right] \geq 0. \quad (9)$$

(c3) *Strike limit*
When $\tau > 0$, $\lim_{K\to\infty} C = 0$ for call options, and $\lim_{K\to 0} P = 0$ for put options.

Proof: Constraint (c3). We have $\lim_{K\to\infty} m = \lim_{K\to\infty} \frac{K}{S_t} = \infty$ for call options, and $\lim_{K\to 0} m = \lim_{K\to 0} \frac{S_t}{K} = \infty$ for put options. Furthermore,

$$\begin{aligned}\lim_{m\to\infty} \sigma_+(b_j^m - me^{w_j^m}) &= \lim_{m\to\infty} \log\left(1 + e^{(b_j^m - me^{w_j^m})}\right) \\ &= \log\left(1 + e^{-\infty}\right) = 0. \end{aligned} \quad (10)$$

(c4) *Boundary conditions*
$(S_t - K)^+ \leq C \leq S_t$ for call options, and $(K - S_t)^+ \leq P \leq Ke^{-r\tau}$ for put options.

(c5) *Expiry value*
When $\tau = 0$, $C = (S_t - K)^+$ and $P = (K - S_t)^+$.

(c6) Constraints (c1), (c3), and (c4) imply that option prices are twice differentiable with respect to K for all $\tau > 0$. Hence, $\frac{\partial^2 C}{\partial K^2} \geq 0$ and $\frac{\partial^2 P}{\partial K^2} \geq 0$.

Proof: Constraint (c6).

$$\frac{\partial^2 y}{\partial K^2} = \frac{\partial^2 y}{\partial m \partial K} = \frac{\partial^2 y}{\partial m^2}\left(\frac{\partial m}{\partial K}\right)^2 + \frac{\partial y}{\partial m}\frac{\partial^2 m}{\partial K^2}.$$

$$\frac{\partial^2 y}{\partial K^2} = \frac{\partial}{\partial K}\left(\frac{\partial y}{\partial m} \cdot \frac{\partial m}{\partial K}\right)$$

$$\frac{\partial}{\partial K}\left(\frac{\partial y}{\partial m}\right) \cdot \frac{\partial m}{\partial K} + \frac{\partial y}{\partial m} \cdot \frac{\partial^2 m}{\partial K^2}$$

$$= \left(\frac{\partial^2 y}{\partial m^2} \cdot \frac{\partial m}{\partial K}\right) \cdot \frac{\partial m}{\partial K} + \frac{\partial y}{\partial m} \cdot \frac{\partial^2 m}{\partial K^2}$$

$$= \frac{\partial^2 y}{\partial m^2} \cdot \left(\frac{\partial m}{\partial K}\right)^2 + \frac{\partial y}{\partial m} \cdot \frac{\partial^2 m}{\partial K^2}$$

$$\frac{\partial^2 y}{\partial m^2} = \sum_{j=1}^{N_h} \left[e^{2w_j^m} \sigma_s'(b_j^m - m e^{w_j^m})\right]\left[\sigma_+(b_j^\tau - \tau e^{w_j^\tau})\right]\left[\sigma_+(b_j^r \pm r e^{w_j^r})\right] \times \left[\sigma_+(b_j^{\sigma_t} + \sigma_t e^{w_j^{\sigma_t}})\right], \quad (11)$$

where $\sigma_s'(x) = \sigma_s(x)(1 - \sigma_s(x)) \geq 0$, thus $\frac{\partial^2 y}{\partial m^2} \geq 0$

$$\sigma_s(x) = \frac{1}{1 + e^{-x}}, \quad \sigma_s'(x) = \frac{e^{-x}}{(1 + e^{-x})^2} \geq 0$$

(c6) Constraints (c1), (c3), and (c4) imply that option prices are twice differentiable with respect to K for all $\tau > 0$. Hence, $\frac{\partial^2 C}{\partial K^2} \geq 0$ and $\frac{\partial^2 P}{\partial K^2} \geq 0$.

Proof: Constraint (c6).

$$\frac{\partial^2 y}{\partial K^2} = \frac{\partial^2 y}{\partial m \partial K} = \frac{\partial^2 y}{\partial m^2}\left(\frac{\partial m}{\partial K}\right)^2 + \frac{\partial y}{\partial m}\frac{\partial^2 m}{\partial K^2}.$$

For call options, $\frac{\partial^2 m}{\partial K^2} = 0$, thus:

$$\frac{\partial^2 y}{\partial K^2} = \frac{\partial^2 y}{\partial m^2}\frac{1}{S_t^2} \geq 0.$$

Likewise for put options:

$$\frac{\partial^2 y}{\partial K^2} = \frac{\partial^2 y}{\partial m^2}\frac{S_t^2}{K^4} + \frac{\partial y}{\partial m}\frac{2S_t}{K^3}.$$

Since dividing a positive constant on both sides of an equation does not change the sign of the equation, we divide $\frac{S_t}{K^3}$ on both side of Eq. (14) and let $\mathcal{F}(y, m) = m\frac{\partial^2 y}{\partial m^2} + 2\frac{\partial y}{\partial m}$ for $K > 0$ and $S_t > 0$. To determine the value of $\mathcal{F}(y, m)$, we approximate it by the second-order Taylor expansion and obtain the following:

$$\begin{aligned}\mathcal{F}(y, m) &= 2y' + my'' \approx f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &= y'(a)2 + y''(a)m \approx f'(a)(m-a) + f''(a)\frac{(m-a)^2}{2}.\end{aligned}$$

Finally, to approximate the value of $\mathcal{F}(y, m)$, we solve two equations $m - a = 2$ and $m = \frac{(m-a)^2}{2}$ and obtain $a = 0$ and $m = 2$. Therefore, $\mathcal{F}(y, m) \approx y(2) \geq 0$. This completes the proof of constraints (c1), (c2), (c3) and (c6). □